Algebra II Quiz Solutions

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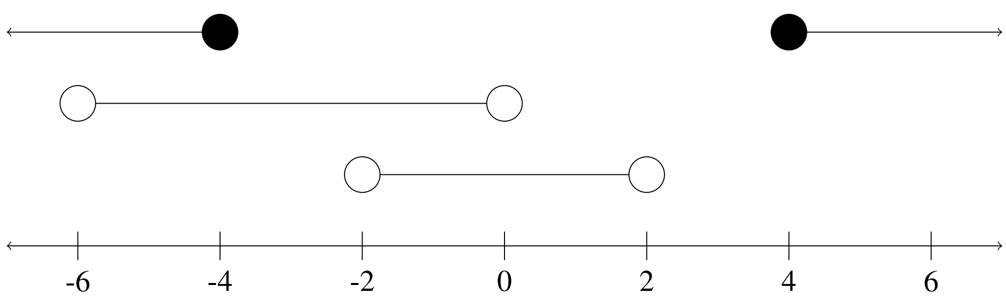
Absolute Values and Inequalities (/40 points)

In the diagram below, I have graphed the solution sets according to 3 different inequalities involving **absolute values**. What are the 3 inequalities that I used? **Please circle your 3 answers.**

$$|x| \geq 4$$

$$|x+3| < 3$$

$$|x| < 2$$



What is the solution set to the equation $|3x+2| = 4x+5$?

$$x = -1$$

What is the solution set to the inequality $|5x-15| \geq 20$?

$x \geq 7$ or $x \leq 1$

What is the solution set to the inequality $|2x-4| \leq -4$?

No solution.

Explanation

In our first encounter with the absolute value function, typically we come to this understanding that it is some function that flips negative values to be positive. However, this understanding fails to grasp at its deeper nature and connection to other mathematics. It is better to understand it as a function that measures the magnitude or distance of some input. In fact, this *is* its meaning in higher dimensions. Here are the definitions of the magnitude function in one, two, and three dimensions.

$$|x| = \sqrt{x^2}$$

$$|(x, y)| = \sqrt{x^2 + y^2}$$
$$|(x, y, z)| = \sqrt{x^2 + y^2 + z^2}$$

If you remember the equation of a circle from geometry class being $r^2 = x^2 + y^2$ and take a look at the 2-dimensional equation, you can see the connection immediately, "a circle is the set of all points with constant magnitude away from its center." Let's use this understanding to push us forward through this problem.

An interval is the one-dimensional analog of a circle. So, just as with 2D circles, all the relevant information to define an interval would be:

- the location of the center
- and the size of the radius

When you want a single interval, you are looking for all points whose distance is *smaller* than the radius. So you will use the the $\leq$ or $<$ sign. This is what is happening for the bottom two equations,

$$|x + 3| < 3$$

$$|x| < 2$$

When you want to be outside of the circle, you want to be at a place *farther* than the radius. So you will use the $\geq$ or $>$ symbol instead:

$$|x| \geq 4$$

The closed dots on the graph indicate equality. So you use $\leq$ or $\geq$ if you have the closed dots.

Sets, Relations, and Functions (/30 points)

State the definition of a **relation**.

A relation between two sets A and B is a set of ordered pairs (a, b) such that a is an element of A and b is an element of B. Commonly, A is referred to as the domain, and B is referred to as the codomain or the range.

State the definition of a **function**.

A function between two sets A and B is a relation between those two sets which also satisfies the condition that: For every element a in A, there exists one and only one ordered pair (a, b) in the relation containing it.

For the following questions, let $A = \{cat, dog, hat\}$ and $B = \{1, 2, 3, 4\}$.
The relations described below have A as their domain and B as their range.

Is this relation a function: $\{(cat, 1), (hat, 2)\}$? **No** The definition of a function states that every element of the domain must appear exactly once in the relation as the first coordinate of some ordered pair. Since in this example, the relation does not contain any dog, it is not a function.

Is this relation a function: $\{(cat, 1), (dog, 1), (hat, 1)\}$? **Yes**

Is this relation a function: $\{(cat, 4), (dog, 2), (hat, 3), (hat, 1)\}$? **No** The definition of a function states that every element of the domain must appear exactly once in the relation as the first coordinate of some ordered pair. Since in this example the relation uses the element, hat, more than once, it is not a function.

Linear Functions (/30 points)

Provide the equation of the line in **Standard Form** that satisfies the following conditions:

- Goes through the point $(3, 4)$
- Is **perpendicular** to the line $y = 2x + 329$

Recall that the Standard Form of some linear equation has the shape, $Ax + By = C$, where A, B, and C are integers. In this problem here, we will begin our work by using the point-slope form. Afterwards, we can transform it to look like the standard form.

We know that we have the point $(3, 4)$ already. So what we need is the slope. In the given equation $y = 2x + 329$, we see that the slope is 2. So if we want a line perpendicular to that, we need to take the negative reciprocal of 2, which is $-1/2$. This gives us the equation, $y - 4 = -\frac{1}{2}(x - 3)$. Rearranging, we get the Standard form which is $x + 2y = 11$.

Provide the equation of the line in **Slope-Intercept Form** that satisfies the following conditions:

- Passes through the x-intercept at 5
- Is **parallel** to the line $2x + 4y = 8$

Again, we will use the point-slope form to get us started. With the first condition, we see that we will use the point $(5, 0)$. With the second, rearranging the equation to get

$$4y = -2x + 8$$

$$y = -\frac{1}{2}x + 4$$

We see that we should use a slope of $-\frac{1}{2}$.

From this, we get an $y = -\frac{1}{2}(x - 5)$. Distributing, we arrive at the answer in slope-intercept form

$$y = -\frac{1}{2}x + \frac{5}{2}$$

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Extra Credit (Score cannot be over 100) (/5 points)

Consider the point $p = (0, 10)$ and the line $y = x/2$. Which point on the line is closest to p, and how far away is that from p?

Intuitively, if I told you to go towards the wall of your room, you would move straight towards the wall at a direction that is perpendicular to it. This is intuitive. Everyone would do it naturally.

In mathematics, this intuition that the path of the shortest distance between a point and a line must be perpendicular to the given line is a trueness. This is a result from Geometry class. It is quite difficult for me to draw diagrams and include them here, so I will avoid describing the proof.

In any case, we know that we must go through the point at $(0, 10)$ and with a slope that is perpendicular to the line $y = x/2$. The slope we want to use is the negative reciprocal of $1/2$ which is -2. From there, we get the line $y - 10 = -2x$. Rewriting, we get $y = -2x + 10$. From here, to get the point on the line that is closest to p, we must find the intersection of the original line with the one we just made. So, we have the system of equations:

$$y = x/2$$

$$y = -2x + 10$$

.

We are looking for a point that satisfies both equations at the same time. The rationale for the next step is as follows: If I have some point that satisfies both equations at the same time, then surely the x-coordinate and the y-coordinate are equal. If that is the case, then I can substitute one representation of y (or x) for the other. If I do this, then I will obtain an equation with only one variable left, and I can deduce the value of that one, just like how I've solved equations before.

So let's do that. I replace the y in the second equation with $x/2$, yielding

$$x/2 = -2x + 10$$

$$5x/2 = 10$$

$$x = 4$$

.

With this $x = 4$, I can put it back into either equation again to yield $y = 2$. So the point closest to $p = (0, 10)$ is the point $(4, 2)$.

To find the distance, we use the Pythagorean theorem between the points $(0, 10)$ and $(4, 2)$ to get

$$\sqrt{4^2 + 8^2} = \sqrt{80}$$

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